Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis I

Mid-term examination — Date : Sept. 9, 2024
Total Marks: 105 — Time: 3 hours
Maximum marks: 100 — Instructor: B V Rajarama Bhat

(1) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying

$$f(f(n)) = n + 2, \ \forall n \in \mathbb{Z}.$$

Show that f is a bijection. [15]

(2) Let $Y = \{3, 4\}$. Let A be the set of all functions from $\mathbb{N}$ to Y. Show that A is uncountable. [15]

(3) Prove the following results using our axioms on the set of real numbers:
(i) If a, b, c are real numbers and $a + b = a + c$, then $b = c$;
(ii) $1 \in \mathbb{R}$ is a positive real number.
[15]

(4) State and prove Archimedean property for real numbers. [15]

(5) Show that every convergent sequence of real numbers is bounded. [15]

(6) (i) Prove Bernoulli's inequality : If $x > -1$,

$$(1 + x)^n \geq 1 + nx, \ \forall \ n \in \mathbb{N}.$$

(Hint: Use mathematical induction).
(ii) Suppose $0 < b < 1$, show that

$$\lim_{n\to\infty} b^n = 0.$$

(Hint: Take $x = \frac{1}{b} - 1$ and apply the previous result). [15]

(7) Suppose $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two sequences of real numbers. Define a new sequence of real numbers $\{c_n\}_{n\in\mathbb{N}}$ by taking

$$c_n = \max\{a_n, b_n\}, \ \forall n \in \mathbb{N}.$$

Prove or disprove: If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are convergent then $\{c_n\}_{n\in\mathbb{N}}$ is convergent. [15]